

Huygens' principle

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Abstract

A short review on Huygens' principle prepared for the "Encyclopedia of Nonlinear Science".

The notion of the *Huygens' principle* (HP) goes back to the classical "Traité de la lumière" ("Treatise on light") by the great Christiaan Huygens published in 1690. In its original meaning HP gives the following geometric description of the wave fronts. Consider a wave front at the moment t as the source of the new (secondary) waves emanating from the points of this front. The HP states that *the new wave front at a later time is the envelope of the fronts of these secondary waves* (see Fig. 1).

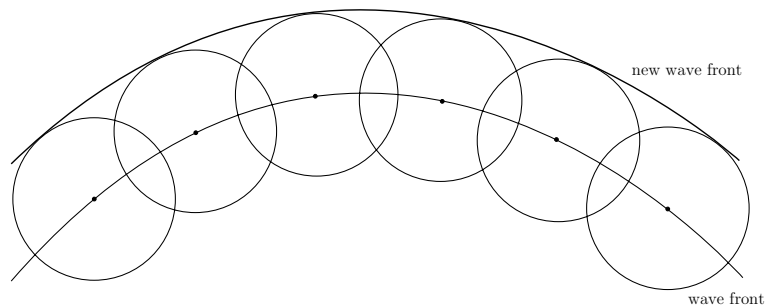


Figure 1

This principle was further elaborated by Fresnel who added the superposition principle for the amplitudes of the secondary waves to explain the phenomenon of diffraction.

In the second half of the XIX-th century various mathematical aspects of the HP were discussed in the pioneering works of Kirchhoff, Beltrami and

Volterra. At the same time it became evident that different authors were using the term "HP" with different meanings. It was Jacques Hadamard who brought clarity in this area.

In his fundamental "Lectures on the Cauchy problem" (Hadamard (1923)) Hadamard introduced the notion of the *Huygens' principle in the narrow sense* ("minor premise"). In physical language one can formulate this principle in the following way: *an instantaneous signal remains instantaneous for every observer at each later time*. This property implies that a localized disturbance will have an effect localized in time at any point, so for example for sound waves after some time there will be complete silence.

Mathematically, the HP in the narrow sense means that the fundamental solution of the corresponding hyperbolic equation vanishes not only outside but also inside the characteristic conoid (and thus must be located on it). Equivalently, a hyperbolic equation satisfies the HP in the narrow sense if the solution of the Cauchy initial-value problem at a point P depends not on all the Cauchy data, but only on its part on the intersection of the characteristic conoid with vertex P with the Cauchy surface.

In contrast to the original principle ("major premise" in Hadamard's terminology), which holds for a general class of wave propagations, the HP in the narrow sense is actually a remarkable property valid only for very special equations. In particular, this never happens in two dimensions, where one has the *diffusion of waves*. One can see it when a pebble falls in water: the front wave it produces will be followed by the so-called *residual waves*, so any object on the water surface will be hit by the subsequent waves many times.

To explain this consider the fundamental solution of the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2}\right)\Phi = 0,$$

$$\Phi(x, 0) = 0, \quad \frac{\partial\Phi}{\partial t}(x, 0) = \delta(x).$$

It has the following form:

$$\Phi = \frac{1}{2\pi} \frac{\theta(t^2 - |x|^2)}{\sqrt{t^2 - |x|^2}}$$

when $n = 2$ and

$$\Phi = \frac{1}{2\pi} \delta(t^2 - |x|^2)$$

when $n = 3$.

Here θ is the Heaviside step-like function and δ is the Dirac delta-function. Thus we see that indeed for $n = 3$ Φ is located on the characteristic cone $t^2 - |x|^2 = 0$, which explains the sharpness of signal transmission in our three-dimensional world, while in two dimensions this is not the case.

Corresponding formulas for the solutions of the Cauchy problem for the wave equation in \mathbf{R}^n for $n = 2$ and $n = 3$ were found by Poisson and Kirchhoff, for the general n it was done by Tedone. From these formulas it follows that for the Euclidean spaces the HP in the narrow sense holds in all odd dimensions starting from 3 and never holds in even dimensions.

Hadamard raised the question of how to describe all second-order hyperbolic equations which satisfy the Huygens' principle in the narrow sense which is known now as *Hadamard problem*. Hadamard found a criterion for this but it was not effective enough to answer this question. For a long time there was a common belief that it holds only for the wave equations in the Euclidean spaces of odd dimensions until 1953 when Karl Stellmacher found other examples (Stellmacher (1953)). He showed that the hyperbolic equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2} + \frac{m(m+1)}{x_1^2}\right)\phi = 0$$

with integer non-negative m satisfies the HP in the narrow sense if the dimension n is odd and large enough: $n \geq 2m + 3$.

Later in 1967-68 Stellmacher and Lagnese solved the Hadamard problem in the class of the hyperbolic equations of the special form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2} + u(x_1)\right)\phi = 0,$$

when the potential depends only on one of the coordinates. They showed that the corresponding potentials are rational and (in modern terminology) can be described as the results of the Darboux transformations applied to $u = 0$ (Lagnese-Stellmacher (1967)).

The first examples of the huygensian potentials $u(x)$ depending on more than one coordinates have been found in 1993 by Berest and Veselov who discovered a close relation of the Hadamard problem with the theory of quantum integrable systems of Calogero-Moser type (Berest-Veselov (1994), see also Chalykh et al (1999) for the latest results in this direction).

For the hyperbolic equations on the manifolds with non-trivial metrics the Hadamard problem is still open even when the number of spatial variables is

three. There are known only several particular cases when HP in the narrow sense is satisfied, including

1) modified wave equation on the symmetric spaces with even multiplicities, in particular on the spheres and simple compact Lie groups (see e.g. Helgason (1994))

2) wave equations on the spaces with the plane wave metrics (Günther (1965)).

For a review of the results on the HP for other relativistic wave equations (e.g. Maxwell's equations) see Günther (1988).

Since the HP in the narrow sense is a very rare phenomenon it is natural to ask about its weaker version. Such a version called *generalized Huygens' principle* was introduced by Lax and Courant and is valid for a general hyperbolic equation. It states that *the singularities of the solution at the point P depend only on the singularities of the initial data and only so far as these data are presented on the characteristic conoid of P* (Courant-Hilbert (1962)). This implies that "in an approximate, and for that matter, usually satisfactory, sense any hyperbolic system does preserve the sharp signals, though in general slightly blurred" (Courant-Hilbert (1962)).

Finally one should mention that the theory of HP is closely related to the deep theory of lacunae in the domains of dependence of hyperbolic equations (Petrovskii (1945), Atiyah, Bott, Gårding (1970, 1973)).

See also Characteristics; Darboux transformation; Gravitational waves; Generalized functions; Quantum nonlinearity.

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